THE ORIGIN OF MAXWELL'S EQUATIONS.

Mike Disney Sept 2015 (added to Aug 2020), 2.7 kw.

1 INTRODUCTION. In the usual form in which they appear to us, with curl **E**'s and div **B**'s, nothing could seem more arbitrary, more unexpected than Maxwell's Equations, the equations which govern Electricity and Magnetism, and hence so much of the modern world. Surely they could only have been discovered by experimentalists stumbling over the weird properties of bits of amber and electric coils? I am here though going to argue the very opposite; that Nature could have organised matters in no other way. Whether this is an original thought I am not positive, but I have not so far come across it in fairly extensive readings.

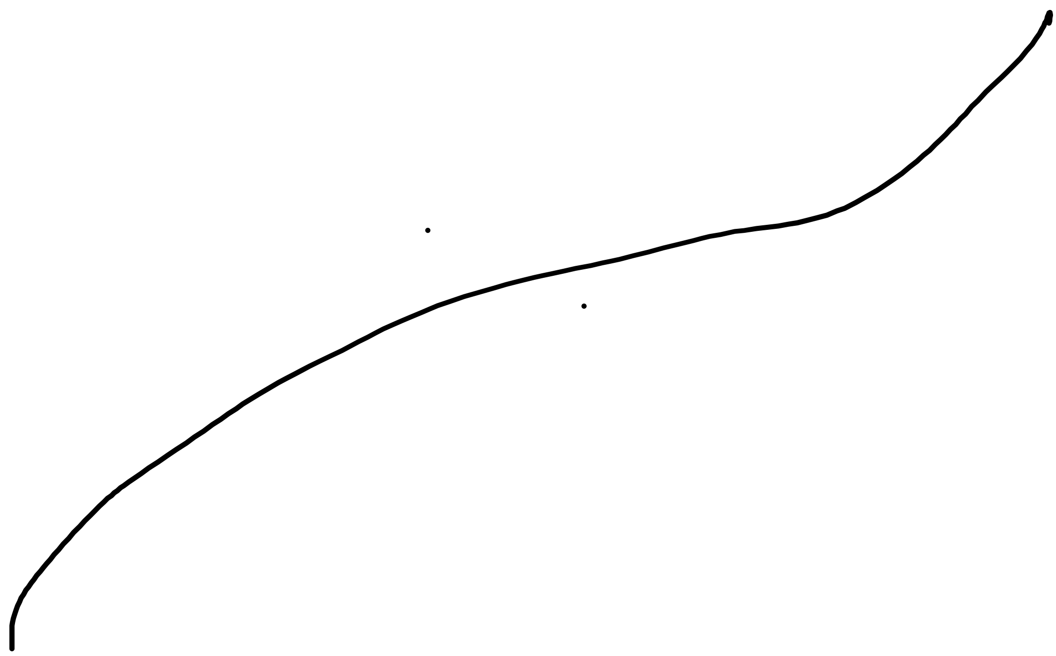
Observations suggest that force-fields are set up in space to which particles (and waves) react. Thus matter sets up gravitational fields which attract other particles nearby. Electric charges set up electric fields which can attract or repel other electric charges, while currents set up magnetic fields in their vicinity which can curve the trajectories of other moving charged particles. These reactions are, we suppose, caused by properties of the fields *local* to the affected particles, not by action at a distance (but see later). And we further believe (observation) that the relationships are usually linear i.e. twice as much cause produces twice as much effect.

To predict quantitative phenomena physicists must therefore look for equations which relate the *local* properties of such force-fields (described by some number φ) to other *local* phenomena e.g. the local density ρ' or the local charge (ρ) or current density $\underline{J}$ [underlined in this case because current is directional (i.e. is what we call a *vector*) while ρ' and ρ are undirected (scalars). Vectors, e.g. velocity, are described by 3 independent numbers, one for each direction in space, whereas scalars are described by only one e.g. Temperature].

2 CURVATURE IN 2 DIMENSIONS; AN ANALOGY

Consider first an analogy, a line curving about in 2 spatial dimensions x and y:

FIG 1



We could describe the curve by relating its vertical y-value at any point to its horizontal x-value as an equation y(x) (e.g. y=x^2). From a physical POV though that is unsatisfactory because neither x nor y are unique axes – one could choose others. The unique local property of a line is its *curvature* – how fast it swings in direction $d\psi$ as you move along it a short distance ds. This curvature ($d\psi$ /ds) is *local and intrinsic to the line* and cannot depend on arbitrarily chosen axes such as x or y.

4 DEL SQUARED

Physical space is 3-dimensional so the question is: "Can we find in 3-dimensions an analogue to the curvature in 2 – which is likewise a local, intrinsic quantity independent of any of the 3 chosen spatial reference coordinates x y and z? It turns out that we can! In any field φ the quantity

$$\Delta^2\varphi \equiv \frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}$$ ['del-squared phi'] is local,

intrinsic and unique in its properties i.e. it is physical as opposed to x, y and z which are merely mathematical, i.e. to some extent arbitrary (i.e. you can change all the axes but get the same value for $\Delta^2\varphi$). $\Delta^2\varphi$ is a measure of by how much the local φ is changed from the average value of φ in its *immediate* surroundings:

i.e. if there is a local minimum in φ, $\Delta^2\varphi$ is positive;

and if there is a local maximum in φ, $\Delta^2\varphi$ is negative;

But if $\Delta^2\varphi = 0$ then φ is spread out as smoothly as possible with no local maximum or minimum.

[Q: Can one show that $\Delta^2\varphi$ is the *only* invariant operator in 3-Space? Or has that been done already?]

4 THE THREE PHYSICAL EQUATIONS

Now it turns out that there are only 3 equations involving $\Delta^2\varphi$ that make any physical sense, so virtually all (linear) physical phenomena must be described by those 3 equations (or some combination of them). This enormously simplifies Mathematical Physics and has all manner of consequences. For instance different theories aimed at trying to explain some physical phenomenon may have to have identical descriptive equations – which means there can be no way of distinguishing between them on the basis of physical measurements alone – because each will predict the same [This was the cautionary point I made in my book *Thinking For Ourselves*, Amazon 2020, sect (15:10), p 461]

Why only 3 equations, and what are they? Recall that $\Delta^2\varphi$ denotes a local minimum, so minus $\Delta^2\varphi$ denotes a local maximum. So far as I can see there are only 3 ways that $\Delta^2\varphi$ *could* relate to the local physics:

I. **It's anchored**. Here there is a local max in φ because it is maintained there by the presence of some other local physical quantity, some other source ρ' existing there. For instance if φ is gravitational field it will be generated (sourced) by the local mass-density ρ' (gm./cc.) in which case:

$$-\Delta^2\varphi \propto \rho'$$

$$or\ \Delta^2\varphi = -\alpha\rho'$$

which is called 'Poisson's Equation . (Eqn I)
α is a universal physical constant associated with this phenomenon, and can be measured once and for all. For example in gravity $\alpha = 4\pi\times$big G .

II **Diffuses away**. If there is a local maximum in φ (Temperature say) without any sustaining source, then stuff (heat say or neutrons) might diffuse away from there so as to even things out. Thus the local φ would fall, a process we would describe by:

$$-\frac{\partial\varphi}{\partial t} \propto -\Delta^2\varphi$$

$$or\ \frac{\partial\varphi}{\partial t} = \beta\times\Delta^2\varphi$$

which is called "The Diffusion Equation" (Eqn II)
where β is another universal constant associated with the phenomenon in question.

III **Attempts to Equilibrate**. If there is a local maximum in φ with no sustaining source ρ' this sets up a local stress which will tend to accelerate φ back towards the norm – until it overshoots like a guitar-string, and then oscillates about equilibrium:

$i.e.$ acceleration: $-\frac{\partial^2 \varphi}{\partial t^2} = -\gamma \times \Delta^2 \varphi$

$or\ \Delta^2 \varphi - \frac{1}{\gamma}\frac{\partial^2 \varphi}{\partial t^2} = 0$ (The "Wave Equation") Eqn III

where γ is again some universal constant that can be measured once and for all by experiment.

One further refinement can be added. If the source of the field is itself directed (e.g. a current $\underline{J}$) then the field $\underline{A}$ it sets up will be directed too (in parallel) and therefore needs to be written as a vector, say $\underline{A}$ [or A_i], so equations I and III can have corresponding vector equivalents

$$\Delta^2 \underline{A} = -\underline{J} \qquad \text{(IA)}$$

$$\Delta^2 \underline{A} - \frac{1}{\gamma}\frac{\partial^2 \underline{A}}{\partial t^2} = 0 \qquad \text{(IIIA)}$$

and these complete our obvious physical-equation set. I can think of no more equations which make physical sense! Can you?

6 ELECTROMAGNETISM

Consider now **Electromagnetism** only. Experiments show that charges ρ and currents $\underline{J}$ set up scalar-fields φ and vector fields $\underline{A}$ respectively. Moreover such fields do not diffuse away as long as the sources ρ and $\underline{J}$ are maintained. So the Diffusion Equation will not apply.

We are then left only with:

Either $\Delta^2 \varphi = -\alpha\rho$

Or $\Delta^2 \varphi - \frac{1}{\gamma}\frac{\partial^2 \varphi}{\partial t^2} = 0$ if ρ=0

And either $\Delta^2 \underline{A} = -\delta \underline{J}$

Or $$\Delta^2 \underline{A} - \frac{1}{\gamma}\frac{\partial^2 \underline{A}}{\partial t^2} = 0 \quad \text{if } \underline{J}=0$$

Whenever the situation is not time varying (i.e. $\frac{\partial^2}{\partial t^2} = 0$)
neither φ nor $\underline{A}$ need be zero [from first and third equations above]. This will be ensured if we add the first two and the last two equations together to get:

$$\Delta^2 \varphi - \frac{1}{\gamma}\frac{\partial^2 \varphi}{\partial t^2} = -\alpha\rho \qquad (1)$$

and $$\Delta^2 \underline{A} - \frac{1}{\gamma}\frac{\partial^2 \underline{A}}{\partial t^2} = -\delta \underline{J} \qquad (2)$$

And these are Maxwell's Equations which describe ALL of Electromagnetism.

That ¾ of the modern world could derive from such threadbare arguments has seemed like a miracle to many. Heinrich Hertz the first experimentalist to grapple with MEs said of them : "One cannot escape the feeling that these mathematical formulae have an independent existence and an intelligence of their own, that they are wiser than we are, wiser even than their discoverers." Nobody has put it better since and, so far as I am aware, nobody has yet found a truly satisfying explanation for this state of affairs. It's as if Nature is telling us "I could act in no other sensible way."

6 **A** AND φ VERSUS **E** and **B.**

Equations (1) and (2) are really 4 independent equations because (2) has 3 components, one for each A_i , in terms of J_i

i.e. $\Delta^2 A_i - \frac{1}{\gamma}\frac{\partial^2 A_i}{\partial t^2} = \delta J_i$

The electromagnetic field is therefore completely specified by 4 quantities only (φ and A_i) – which is why you need 4 independent equations to solve for it completely. Note that the equations are nearly all identical (apart from the source terms ρ and J_i) which is highly convenient from a mathematical POV because once you find a way to solve one you can likewise solve all 4.

Most of us first encounter MEs in terms of the more familiar Electric and Magnetic fields **E** and **B** which are related to φ and $\underline{A}$ by:

$$\underline{E} = -grad\varphi - \frac{\partial \underline{A}}{\partial t}$$

and $\underline{B} = curl\underline{A}$

and $div\underline{A} = -\frac{1}{\gamma}\frac{\partial \varphi}{\partial t}$

when ME's, or more rightly the Maxwell-Heaviside Eqs. become:

$$div\underline{E} = \alpha\rho$$

$$curl\underline{E} = -\frac{\partial \underline{B}}{\partial t}$$

$$div\underline{B} = 0$$

$$curl\underline{B} = \delta \underline{J} + \frac{1}{\gamma}\frac{\partial \underline{E}}{\partial t}$$

Personally I like φ and **A** in preference to **E** and **B** because: (a) **A** is parallel to its source term J_i;(b) **E** and **B** between them have 6 components, which is 2 more than the 4 in MEs. Thus **E** and **B** *cannot be independent* fields.[On the other hand **A** and φ are hard to measure, and indeed not unique: the so called 'gauge-problem'.]

7 CHARGE CONSERVATION

Most modern textbooks invoke charge conservation to derive MEs. But we haven't needed that and I don't believe it should be used on either logical or historic grounds. MEs describe the fields which arise from sources, not the properties of the sources themselves, which charge conservation does. In any case I don't believe charge conservation was known in Maxwell's day. So we shouldn't need it. However if, for some reason, we insist on conservation then we must introduce the Continuity Equation:

$$\frac{\partial \rho}{\partial t} + div\underline{J} = 0$$

Now take the div of ME 2 and

$$div\ curl\ B = \delta div\underline{J} + \frac{1}{\gamma}\frac{\partial}{\partial t}\left(div\underline{E}\right)$$

$$so \quad 0 = \delta div\underline{J} + \frac{1}{\gamma}\frac{\partial}{\partial t}\left(\alpha\rho\right)$$

$$or \quad \frac{\partial \rho}{\partial t} + \frac{\delta\gamma}{\alpha} div\underline{J} = 0$$

$$\text{So} \qquad \delta = \frac{\alpha}{\gamma} \qquad \text{(IV)}$$

Which reduces the number of constants in MEs to 2 but does not change their form. This seems to underline my point that Charge Conservation is a separate physical law unnecessary to MEs. I believe Wigner found a profound argument as to why charge should be conserved, but it came long after Maxwell's time.

9 RELATIVITY

Of many extraordinary and unforeseen outcomes of MEs Relativity is one of the most remarkable. MEs force us into Relativity because Maxwell's constant γ has the nature of a universal velocity-squared (purely on dimensional grounds. See * argument below). In fact measurement shows $\gamma = c^2$ where c is the speed of light![Kohlsrauch and Weber had measured alpha and gamma before Maxwell devised his equations and he made use of their measurements to show that $\gamma = c^2$]

But velocities (Newton) are supposed to be *relative* and so should not appear in a general law of Nature. But here is one that does, c the speed of light! Special Relativity consists entirely in the modifications to Newtonian Physics required to accommodate this wholly unexpected outcome of MEs. Personally I do not see how anyone who has not first been introduced to MEs has any chance of appreciating the whys and wherefores of Relativity.

[*Here is the argument: If we look at Eqn. (IIIA) above, i.e. :

$$\Delta^2 \underline{A} = -\underline{J} \qquad \text{(IA)}$$

$$\Delta^2 \underline{A} - \frac{1}{\gamma}\frac{\partial^2 \underline{A}}{\partial t^2} = 0 \qquad \text{(IIIA)}$$

then the right hand side has no dimensions, so neither can the left. The dimensions of the A obviously cancel and from its definition above i.e:

$$\Delta^2 \varphi \equiv \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2}$$

$\Delta^2\varphi$ without the phi obviously has the dimensions of 1/(distance squared). So for (IIIA) to be true γ must have the dimensions (distance squared) over (time squared) i.e. velocity squared. (measurement shows that $\gamma = c^2$ where c turns out to be the speed of light); Is there some argument for why that should be so? That could be very important.]

8 QUESTIONS:

(a) How can you solve for the 6 unknowns E_i *and* B_i with only 4 eqns ? Because the 2 curl equations are vector equations and thus have 3 components each. In other words there are 8 eqns for 6 unknowns.

(b) Why do we insist on a field theory, instead of action at a distance? You can have (transient) currents in open circuits, which is why you need the 'displacement current' term in MEs. Imagine an AC generator driving a simple 2-plate condenser. If you can think in terms of 'action at a distance' then you can explain what is going on as simply charge building up and falling away on the opposed condenser plates – as I was taught it at school. But if you insist on a local field theory then that implies polarisation of the vacuum between them – which is precisely why Kelvin and his contemporaries rejected the Faraday/ Maxwell interpretation. So why do we welcome it now? What has changed in the interval? Has it got something to do with insisting on a finite speed of propagation?

(c) This derivation is akin to the dimensional analysis derivation for the simple harmonic pendulum equation, and the derivations of the equations of elasticity and of Einstein's Field Equations. They are all based on the argument that nothing else simpler will work. They are thus satisfying from a mathematical POV but are they satisfying physically? I find a SHM argument for the pendulum equation much more satisfying than dimensional analysis, even though that analysis cannot be gainsaid.

(d) Is any of what I have said original? And if not then why isn't this explained up front in all Electricity and Magnetism textbooks? Or am I haunted by some silly mistake which everyone else can see and I can't? Note that this same argument also implies that Gravitation MUST HAVE the inverse square form. Did everybody know that already, except me? [According to Weinberg the exponent could take *any* value, not necessarily 2.]

THE FASCINATING BACKGROUND STORY

Maxwell is treated like a 'genius' nowadays, but, as always, things were more complex and thus far more interesting. His mentor Lord Kelvin, and pretty nearly everyone else, thought MEs were nonsense because they implied the polarization of a supposedly structureless vacuum; and one can understand why they argued so. Even Maxwell probably had doubts himself, because he made very little of them thereafter, did no experiments, and relegated them to a single chapter in the second Volume of his huge treatise on Electricity and Magnetism, which came out in 1873. Hertz, so I read somewhere, had never heard of them before he started on his own work in Karlsruhe in 1887, by which time Maxwell was dead (1879); only Helmholtz seemed to think they should be followed up and it is remarkable that Hertz was his former student. And the propagation of radio waves had already been demonstrated dramatically by David Hughes in London back in 1870, 17 years before Hertz, when he discovered serendipitously that sparks from his lab. in Great Portland Street propagated detectable electric signals for over 500 meters down that thoroughfare. Disastrously he called in the prestigious Royal Society to witness that miracle and they sent a botanist, a zoologist and a mathematician (Stokes) to do so. In one of the greatest blunders in scientific history Stokes pooh-poohed the demonstration as "Nothing more than Induction", [which it so obviously was not, because Induction falls off with the 4^{th} power of the distance.] Then there was Maxwell's maths, which puzzled everybody; he called his Curls 'components of a

Quaternion', which hardly any scientist had heard of at the time, and which were quickly dropped as 'clumsy' Then a hero stepped in ; the innumerate 16 year old telegraph operator Oliver Heaviside who found Maxwell's tomes in his local library in London (1873). He couldn't understand a word, but gave up his job and lived for years on potatoes while he tried to find out why telegraphy was so slow (then a few words a minute). He invented Vector Calculus, recast MEs using that, and discovered that Lord Kelvin had made a huge blunder in his Theory of Telegraphy by omitting Induction. Heaviside then showed how to speed telegraphy up by thousands, and also showed how radio waves could curve around the Earth if there was a hypothetical 'ionosphere', which wasn't actually found until 1924 by Appleton. Heaviside died, unheralded, probably of malnutrition. N.B. This is a great story, never properly told. I stumbled upon it while cliff-walking along the coast near Cardiff in 1996 with my friend Clinton Osborn; we discovered a plaque in the crumbling stone wall of a tiny old church commemorating the first radio message sent across water (by Marconi from there, 10 miles across the Bristol Channel, in 1897). We've tried in various ways to tell the true story of radio but the myth has so much momentum that nobody wants to hear the truth now; it's like the myth of the Wright Brothers making the first powered flight in 1903, even though one can prove mathematically that they could never have done so. Fake news has a long history, and so has fake history, as I've tried to show in my *"History of The Brits*" Amazon, 2020.