THINKING FOR OURSELVES Michael Disney

EXERCISES WITH ANSWERS Draft 14/1/20 (3.5 k-words)

NB: If you learn to do all these exercises they should, in the long run, pay back what you spent on this book many many times over. Try hundreds, or even thousands.

EXERCISES (4:1) ODDS AND PROBABILITIS

1 If P(H)= 0.2 what is P($\bar{H}$)?{Hint: use Eqn. (4:1)}
(Answer 0.8)

2 From the results of Q1 what is O(H)? { Hint: use Eqn.(4:2)}
(Answer ¼)

3 From the results of Q2 what is O($\bar{H}$)?{Hint use Eqn. (4:3}.
(4:1 or 4)

4 What are the Odds *on* the H of Q1?
(1:4 or ¼)

5 What are the Odds *against* the H of Q1?
(4:1 or 4)

6 If P(H|E) = 0.4 what is P($\bar{H}$|E)? {Hint: use Eqn. (4:1)}
Answer (0.6)

7 What is O(H|E) for the H in Q6?
(0.4/0.6 = 2/3 = 2:3, i.e. 2 to 3 *on* or 3 to 2 *against*).

8 If P($\bar{H}$) = 0.2 what are P(H), O(H) and O($\bar{H}$) ?
(0.8, 4, ¼)

9 If P(H|E) = 0.1 what are the Odds *on* H i.e. O(H|E)?
(1/9).

10 In Q9 what are the Odds *against* H?
(9:1 or 9)

EXERCISES (5:1): THE DETECTIVE'S EQUATION.

1 If W(E_1 |H) = 2 and W(E_2 |H) = 6 what is O(H|,E_1,E_2) if the Prior O(H) =3 ?
(Answer: $2\times6\times3$ = 36)

2 There are 3 clues with Weights 4, 3 and 1/6 bearing on Hypothesis H. If your Prior O(H) = 8 what are your Posterior combined Odds on H?
($4\times3\times1/6\times$ 8 = 16 i.e. 16 to 1 on)

3 There are 2 clues with Probabilities P(E_1|H) = 0.6 and P(E_2|H)= 0.5 while your Prior O(H) is 5 to 1 on. What are your Posterior Odds on H? {Hint: *If* the evidences E_1 and E_2 already exists (they do here) then it must be true [See Eqn. (4:1)] that P(E_1|H) +P($E_1|\bar{H}$) = 1 so one can estimate the P(E|$\bar{H}$) s from the P(E|H)s, and hence the W(E|H)s from Eqn. (4:5). Thus one can eventually arrive at the Posterior Odds using the Detective's Equation.}
(P($E_1|\bar{H}$) = 0.4; P($E_2|\bar{H}$) = 0.5; thus W(E_1|H) = 0.6/0.4 = 3/2 and W(H|E_2) = 0.5/0.5 = 1 and so the Detectives Equation yields Posterior odds of 3/2 $\times$ 1 $\times$ 5 = 15/2 or 15 to 2 on) .

4 A detective is trying to solve a serious crime and his hypothesis is that X is guilty. His incomplete 6-column Inference Table looks like this:

Clue	O(H)	P(E|H)	P(E|$\bar{H}$)	W(E|H)	O(H|E)
	1/3 Prior				
1		0.25			

2		0.8			
3		0.6			
4		0.9			

Complete his Inference Table and hence arrive at his Posterior Odds at the bottom right hand corner of his table.

{Hint: you can calculate the $P(E|\bar{H})$s from the P(E|H)s as we did in the last question using Eqn. (4:1), and hence the Weights using Eqn.(4:5) and then use Bayes' Rule (4:5) to update the Posterior Odds in the last column line by line.}
(Answer:

Clue	O(H)	P(E|H)	$P(E\|\bar{H})$	W(E|H)	O(H|E)
	1/3 Prior				
1		0.25	0.75	1/3	$1/3\times 1/3 = 1/9$
2		0.8	0.2	4	$4\times 1/9 = 4/9$
3		0.6	0.4	3/2	12/18 = 2/3
4		0.9	0.1	9	6

5 Your rather nice lawn-mower disappears from your garden and a couple of weeks later you notice your new neighbour mowing his lawn with a machine that looks suspiciously like yours. Inviting yourself round you find that it is indeed an identical model, though they are rather common. However the distinctly dented grass-bin of your machine has been replaced by a brand new one. Your neighbour boasts that their teen-age son had bought the mower in a boot-sale as a birthday present for his father. Both parents are doctors, and so very wealthy. They had kindly invited you to their house-warming party and

seemed very amicable and decent people. What are the Odds that that they have stolen your mower? Identify your clues and construct a 4-column Inference Table for your combined Odds on their guilt. Be prepared to defend your individual Weights and your Prior on their guilt.
(There is no correct answer. But when I use a Prior of 1/20 and put in my best guesses at the Weights of 6 different clues I reach Odds on guilt of 12 to 1. But are those Odds good enough to act on?)

6 Repeat the exercise of Q5 but use only the Weights permitted in Section (5:7).
(If I start from a Prior of 1/16 I finish with Odds on guilt of 2:1, far too weak to conclude anything.)

7 Repeat Q6 but start from a Neutral Prior.
(I find Odds on Guilt of 32 to 1 on. This example gives much food for thought. In particular what would be a reasonable Prior? A neutral Prior sounds fine until you realize that it implies that half of all neighbours are common thieves. And why should one use Weights drawn from Section (5:7) only? Were you convinced by the arguments therein?).

EXERCISES (9:1) SCATTER AND STANDARD DEVIATION

1 Mrs Jones' corner shop is open between 8 am and 6 pm. It's pretty busy and so she thinks of employing an assistant for a few hours a day – but during which hours? So one day she records the number of customers x_i arriving in each 1-hour interval and finds:

	8 am:	9	10	11	12	13	14	15	16	17
Number:	20	25	17	31	42	33	21	19	25	17

(i)Work out the average number $\bar{x}$ arriving per hour.

(ii) Calculate $(x_i - \overline{x})$ for each interval.

(iii) Square the last number in (ii) and accumulate your answers: $\sum_{i=8}^{i=17}(x_i - \overline{x})^2$.

(iv) From your answer to (iii) and the number of hours in the day calculate the scatter from the formula on p (9:4) and hence the standard deviation σ_x from Eqn.1 in Chapt. 9. Keep your answers for later.

((i) = 25, (iii) = 686. (iv) 68.6 and hence σ_x = 8.28).

EXERCISE (9:2) VERTEX NUMBERS

Reconstruct the entire branching diagram of Fig (9:1) [sometimes called 'Pascal's Triangle] for yourself. Doing so should help you to get a feel for why middling numbers of Heads , around the average, are much more likely [more 'Probable'] to turn up than extreme numbers such as Zero (or All) Heads and hence where the later important diagrams such as Fig (9:2) and in particular the Bell Curve Fig (9:3) come from. It's rather obvious really, but mathematicians [Statisticians in particular] can make it all seem fiendishly complicated [sometimes I think by design], and then use it to argue that we have to rely on them to draw sound conclusions from numerical data. That's nonsense – as we shall come to see more clearly in Chapt. 11.

EXERCISES (9:3) CALCULATING PROBABILITIES

1 Using your own branching diagram [or mine Fig (9:1)] to calculate the Probability of exactly 3 Heads turning up in 4 tosses of a fair coin. {Hint: look under 4 tosses until you get down to 3 Heads.}

(There are altogether 2^4 = 16 possible Combinations of Heads and Tails that could be thrown. Of these, according to the diagram only 4 [the vertex number under 3 Heads] contain exactly 3 Heads. Thus 4 out of 16 Combinations contain 3 Heads: thus the Probability of getting 3 heads in 4 tosses = 4/16 = ¼ = 0.25 = 25% .)

2 What is the Probability of getting exactly 4 Heads in 6 tosses? {Hint: use vertex diagram}
($15/2^6$ = 15/64)

3 What is the Probability of getting 4 *or more* Heads in 7 tosses? {Hint: Add up the vertex numbers for 4, 5, 6 and 7 Heads and divide their sum by 2^7 − the total number of possible Combinations when there are 7 tosses.
(35+21+7+1 = 64, divide by 2^7= 128 to get 0.50 or about ½)

4 What is the Probability of getting just 1 Head in 9 tosses?
(see vertex diagram: 1/ 2^9 =1/512 = .02)

5 What is the Probability of getting 4 Heads in 9 tosses?
(From vertex diagram = $126/2^9$=126/512 or about 25 %)

6 What is Probability of getting 4 or 5 Heads in 9 tosses?
(From vertex diagram = (126+126)/ 2^9=252/512 or about 49%).

EXERCISES (9:4) EXPECTED SCATTER

1 Toss a coin 20 times and record the number of Heads which turn up. What is the mean or average number you would expect and the expected standard deviation in that number? {Hint use Eqn. (3)}

2 Repeat Q1 a dozen times altogether. How do the actual number of Heads per trial scatter about the expected number of 10 Heads? About 2/3rd of the values *should* lie within plus or minus one standard deviation of 10. Do they?

3 How many Heads turned up altogether in your 12 times 20 = 240 tosses. How many would you expect and what would be your expected standard deviation in that number?
(Expect 240/2 = 120 Heads with an expected standard deviation of $\sqrt{240}/2$ or about ±8 . The Odds are 2 to 1 that you will find a value lying between 112 and 128 [i.e. within one standard deviation of the expected value] and 20 to 1 a value between 104 and 136 [i.e. within 2 standard deviations of the expected value 120]. Values outside this last range are rather unexpected and suggestive, but by no means conclusive, evidence of an unfair coin.)

EXERCISES (9:5) NUMBERS SIGNIFICANT AND INSIGNIFICANT

1 A roulette wheel can come up either Red or Black. You observe that in 400 spins of the wheel at your local casino Red comes up more often than Black 30 times. Is that good reason to bet preferentially on Red in future?
{Hint: estimate the Expected number of Reds and the expected standard deviation σ in that number. Then consult Table (9:2).}
(According to Eqn (3) the expected number of Reds is Np = 400 times 0.5 = 200 with an expected standard deviation of $\sqrt{400}/2$ = ±20/2 = ±10. The actual number of Reds must be 200 + 30/2 = 215 which is 15 more than expected but only 15/10 standard deviations more. According to Table (9:1) this is not yet a 2-sigma result so it implies 'Insignificant, ignore'.)

2 How large would the discrepancy between Reds and Blacks have to be to suggest the wheel is biased? {Table (9:2)}
(from the answer to (1) 2 standard deviations, which [according to table (9:2] would be strongly suggestive,

would have to be 2 times 10, or 20 more than the expected value of 200 Reds. So you would need 220 Reds and therefore only 180 Blacks, which is a total discrepancy of 40.)

3 So how large would the discrepancy have to be to leave you almost certain the wheel is biased? {Hint:Table (9:2) }
(You need a 3-sigma result with at least 3 times 10 more Reds than the expected value of 200. So the discrepancy between Reds and Blacks would have to be 60 or more.)

4 Your financial adviser points out that of its 28 major purchases of stocks last year by Hedge Fund X no less than 19 did better than the stock market average. Is that good reason to invest your money with X?
{Hint: think averages and standard deviations then consult Table (9:2) }
(Answer: On average you would expect a random fund to pick $28/2 \pm \sqrt{28}/2$ or 14 ± 2.7 'winners' or better than average stocks [see Sect (9:2)] where 2.7 is 1 sigma σ. For a better than 2 sigma positive, i.e. significant, result it would need to pick more than 14 + 2 times 2.7 or at least 20 winners. 19 is only a 1 sigma result and thus according to Table (9:2) 'insignificant'. So the figures are not a significant inducement to buy X. There are however wider issues involved here. If there were enough Hedge Funds some of them will do significantly better than average by pure chanc*e*. Indeed according to de Moivre [see Table(9:1)] there is a 5% Probability (.05) of a fund's results lying outside 2 sigma and thus a 2.5% of its lying to the good side of the +2 sigma boundary *by pure chance*. Thus if there were 1000 Hedge funds to pick from about 25 of them would look 'attractive' in any one year. But that would be no good reason to buy them because they would be no more than illusory 'false-positives' (a nasty problem with drug trials in particular). What about a + 3 sigma performance? According to Table (9:1) the Probability of that happening by chance is half of 0.003 or 0.0015 or about 1 in a thousand. Even so one has to be very

careful. Why on Earth should Hedge fund performances have a 'Normal Distribution' conforming to de Moivre's beautiful bell shaped curve and therefore to his Probabilities? No reason that I know of. All the evidence shows anyway that past performance at such picking of stocks is no indicator of future performance. It seems to be nearly all luck. Even so millions are induced to invest billions in all sorts of questionable funds. A typical Hedge-Fund manager will charge you heavy fees, and still cream off 20% of any profits. Some make billions a year for doing nothing useful. This is part of the unspoken cost of capitalism. The general point for us here is that it all too easy to be bamboozled by numbers, because we have no animal instinct for dealing with them − after all numbers were only invented about 5000 years ago for shop-keeping and taxes. Numbers have to be turned into *Weights* before they can be used in a serious argument, and that, as we shall see, is tricky. All of us must be able to estimate which numbers might be significant, and to dismiss the rest.)

EXERCISES (9:6) CORRECTING INFERENCE TABLES

1 Fill in all the blank cells for Clues 1 to 6 below:

INFERENCE TABLE ONE

(1)	(2)	(3)	(4)	(5)	(6)
Clue	W(E|H)	O(H|E)	Q	$2^{\pm\sqrt{Q}/2}$	O(H|E)
Prior		1			(corrected)
1	2^4	2^4	4	$2^{\pm 1}$	$2^{4-1} = 2^3$
2	2^2	2^6	6		
3	2^{-1}		7		
4	2^3				
5	2^5				
6	2^{-6}				

INFERENCE TABLE ONE (Answers}

(1)	(2)	(3)	(4)	(5)	(6)
Clue	W(E|H)	O(H|E)	Q	$2^{\pm\sqrt{Q}/2}$	O(H|E)
Prior		1			(corrected)
1			4	$2^{\pm1}$	
2		2^6	6	$2^{\pm1}$	$2^{6-1}=2^5$
3		2^5	7	$2^{\pm1}$	$2^{5-1}=2^4$
4		2^8	10	$2^{\pm2}$	2^6
5		2^{13}	15	$2^{\pm2}$	$2^{13-2}=2^{11}$
6		2^7	21	$2^{\pm2}$	2^5

2 Do the same for Inference Table Two:

INFERENCE TABLE TWO

(1)	(2)	(3)	(4)	(5)	(6)
Clue	W(E|H)	O(H|E)	Q	$2^{\pm\sqrt{Q}/2}$	O(H|E)
Prior		1			(corrected)
1	2^{-4}				
2	2^{-2}				
3	2				
4	2^{-3}				
5	2^{-5}				
6	2^6				

And the answers are in the Table below:

INFERENCE TABLE TWO (Answers)

(1)	(2)	(3)	(4)	(5)	(6)
Clue	W(E|H)	O(H|E)	Q	$2^{\pm\sqrt{Q}/2}$	O(H|E)
Prior		1			(corrected)
1		2^{-4}	4	$2^{\pm 1}$	$2^{-4+1} = 2^{-3}$
2		2^{-6}	6	Ditto	2^{-5}
3		2^{-5}	7	Ditto	2^{-4}
4		2^{-8}	10	$2^{\pm 2}$	2^{-6}
5		2^{-13}	15	Ditto	2^{-11}
6		2^{-7}	21	Ditto	2^{-5}

3 Do the same for Inference Table Three, which is only the bottom of a much larger table with 26 clues in all.:

INFERENCE TABLE THREE

(1)	(2)	(3)	(4)	(5)	(6)
Clue	W(E|H)	O(H|E)	Q	$2^{\pm\sqrt{Q}/2}$	O(H|E)
					(corrected)
21	2^{4}	2^{8}	46		2^{5}
22	2^{2}				
23	2^{-2}				
24	2				
25	2^{-3}				
26	2^{4}				

INFERENCE TABLE THREE (Answers)

(1)	(2)	(3)	(4)	(5)	(6)
Clue	W(E|H)	O(H|E)	Q	$2^{\pm\sqrt{Q}/2}$	O(H|E)
					(corrected)
21		2^8	46	$2^{\pm 3}$	2^5
22		2^{10}	48	Ditto	2^7
23		2^8	50	$2^{\pm 4}$	2^4
24		2^9	51	Ditto	2^5
25		2^6	54	Ditto	2^2
26		2^{10}	58	Ditto	2^6

EXERCISES (11:1) PERMUTATIONS AND COMBINATIONS.

(For these exercises it would help, but it is not essential, to have a calculator with factorial button usually denoted 'x!')

1 A people-mover can seat 7 passengers. In how many different arrangements (permutations) can they be seated?

{Hint: use Equation (5).}

(There are 7 places for each passenger to sit so we want to know how many permutations there are of 7 people. That according to Eqn (5) is:

Perms (7 from 7) = 7 !/ (7-7)! = 7!/ 0! = 7!/1. You might be surprised by this last step where 0! turns out to be 1, but so it has to be to always make sense. So the answer is 7! or 5040 different permutations. Notice how quickly factorials mount up. Thus 12! = 479,001,600 or nearly half a billion! Even calculators soon give up.)

2 A fair coin is tossed 8 times. In how many different orderings (combinations) can 5 Heads turn up?

{Hint : use Eqn.(6)}

(Combs (5 from 8) = 8!/5!(8-5)! = 40,320/120 × 6 = 56).

3 Seven people want to get into a 4-seater car. How many combinations of 4 people could be seated. {Use Eqn (6)}

(Combs 4 from 7 = 7!/3! $\times$4! = 35)

4 How many different combinations of 4 Hearts are there in a pack of 52 playing cards?
{Hint: use Eqn. (6)}
(There are 13 Hearts so Combs 4 from 13 = 13!/(13-4)!4! = 13!/9!4!= 13$\times$12$\times$11$\times$10/4! = 17,600/24 = 715)

5 Fifty passengers are to be seated in a small 50-seater airliner. In how many different ways (permutations) can they be seated?
(If you use Eqn.(5) the answer is 50!/(50-50)! = 50!/ 1. But 50! is a vast number beyond the exact calculation of even my powerful calculator which can only make a roughish estimate of about 3 times 10 multiplied by itself 64 times over. But when Statistics was first invented such numbers had to be laboriously calculated. So a new occupation came into being, humble drudges called 'Computers' who were employed to carry out the tediously mind-numbing calculations. There were two (un?) predictable consequences. The computers escaped from their stools, puffed themselves up in jargon, and called themselves 'Statisticians'. And countless numbers of Statistics text-books were produced consisting mostly of tricks (approximations) to avoid calculating large factorials. You can't blame the poor devils can you? But you don't have to believe them. People will do anything to get out of a tedious job. Wouldn't you? As one Statistician put it: "Statisticians have already over run every branch of science with a rapidity of conquest rivalled only by Attilla, Mohammed and the Colorado beetle.")

EXERCISES (11:4) THE GAMBLER'S SECRET.
(You will certainly need a calculator to do these)

1 What are the chances [i.e. the Probability] of throwing exactly two sixes in 5 throws of a dice? {Hint: use Eqns. (7) and (6) }
(P(2,5) = No. of Combs. of 2 from 5 × $p^2 \times (1-p)^{5-2}$ where the Probability p of throwing a six is obviously 1/6 (6 faces)
so $P(2,5) = \frac{5!}{3! \times 2!} \times \left(\frac{1}{6}\right)^2 \times \left[\frac{5}{6}\right]^{5-2}$ = 0.161 [16%)])

2 What are the chances of throwing more than 3 sixes in 5 throws of a dice?
(Simply add up P(4,5) and P(5,5) the Probabilities of throwing 4 and 5 sixes using Bernoulli's Eqn. (7) in each case. The answer is 0.00335 or about 1/300)

3 What is the Probability of drawing 6 picture cards in a hand of 13 cards?
{Hint: use Eqns. (7) and (6)}
(There are 12 picture cards out of 52 so the Probability of drawing any one is 12/52, and the Probability of drawing 4 out of 13 is therefore, by Bernoulli's Eqn. (7)`:

$$P(4,13) = \left(\frac{13!}{9!4!}\right) \times \left(\frac{12}{52}\right)^4 \times \left(\frac{40}{52}\right)^{13-4}$$ =0.132 or 13%)

4 The casualty rate among RAF Bomber Command Blenheim aircrew during WWII was 9% per sortie. What were the chances of a crew surviving a Tour of Duty of 30 sorties?
{Hint: As usual use Eqn. (7), Bernoulli's equation for gamblers }
(We can calculate P(30,30) where the Probability of surviving one sortie was obviously [1-(9/100)]= 0.91. If we now use Eqn.(7):

$$P(30,30) = \frac{30!}{0!30!} \times (0.91)^{30} \times \left(\frac{9}{100}\right)^0 = 1 \times (0.91)^{30} \times 1 = 0.06$$ a

very sad 6%. Most of those airmen were being sent to their deaths. Fortunatly the Blenheim was eventually replaced by much better aircraft such as the Wellington. But nearly all bombers, on all sides, were very dangerous aircraft to fly in then. One Luftwaffe night-fighter crew claimed to have shot down more than 500 Lancasters, each with 7 aircrew, who rarely escaped through its hopelessly small hatches. In my opinion certain RAF Air Marshalls should have been shot for criminal manslaughter, because they knew what was going on. Why weren't Lancaster crews turning up in German POW camps like our other bomber crews?)

5 If you like an occasional flutter, as I do, how do you know the bookie is going to treat you fairly? Many don't. [Bookmaking is a very lucrative business. For instance one on-line bookmaker in Stoke paid tax on over £250 million pounds of annual earnings.] Anyway if you know the betting Odds O_j on offer against each and every horse j in the race it should be possible to work backwards and find what fraction f of all the money that is bet that bookies plan on returning to their punters. Since we've learned all this stuff on Odds why don't we have a go at working it out?

PROBLEM: (Hard). Show that if each winning punter gets back his original wager plus O_j times that wager, where O_j are the Odds on offer against horse j winning (N horses in the race altogether) then:

$$\frac{1}{f} = \sum_{j=1}^{j=N} \left(\frac{1}{1+O_j} \right) \quad \text{(a)}$$

where Σ sums up the contents of the bracket for all the horses in the race.

APPLICATION(much easier): Go to the racing page, pick a race, and look at all the individual Odds on offer. Then, either by hand, or using a calculator, calculate f for that race using Equation (a) above. Repeat for several races. What conclusions do you draw, and will you continue to bet in future? When I did this I got a shock. A little maths can sometimes save you a lot of money.